Application of A level Mathematics and Further Mathematics

This application makes use of the following:

Topics from A level Mathematics	-	Derivative of e^x
	-	Trigonometry
Topics from Mechanics 1	-	Newton's second law
Topics from AS Further Mathematics	-	2 x 2 matrices
Topics from A2 Further Mathematics	-	Solving second order linear differential equations using an auxiliary equation
	-	Eigenvalues and eigenvectors of a matrix

Coupled Spring System

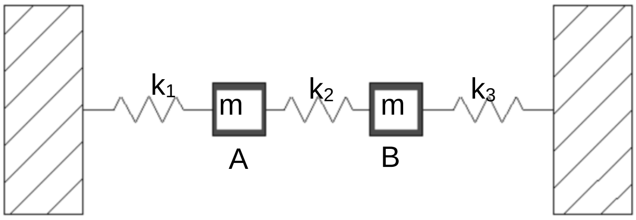
Newton's second law can be used to set up equations for the motion of a pair of masses connected by springs. Hooke's law is used to find the tension in the springs. The system of simultaneous equations can be written using matrices, which make the analysis of the vibration of the masses more straightforward. The eigenvectors of the matrix determine the stable modes of vibration of the system.

A vehicle suspension system could be modelled using a similar mass and spring system.

The problem:

Consider the simple system shown of 2 masses and 3 springs. The masses are constrained to move only in the horizontal direction between two fixed walls.

What are the normal modes of oscillation?



Assumptions:

The tension in the springs is given by Hooke's law, $T = kx$, where k is a constant related to the stiffness of the spring and x is the displacement from the equilibrium position.

The system is set up with all three springs at their natural length.

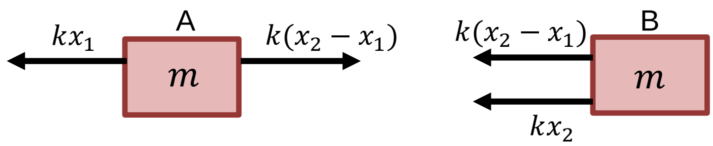
Assume the masses A and B are equal and the stiffness of the springs are equal, $k_1 = k_2 = k_3 = k$.

Assume there is no damping in the system and therefore the amplitude of vibration remains constant.

Setting up the equations:

At time t, let displacement of A be x_1 , and displacement of B be x_2 .

Force diagrams show the tensions in the springs



Using Newton's 2nd law, (acceleration is given by $\ddot{x}$)

For A $\quad m\ddot{x}_1 = k(x_2 - x_1) - kx_1 \quad = -2kx_1 + kx_2$

For B $\quad m\ddot{x}_2 = -k(x_2 - x_1) - kx_2 \quad = +kx_1 - 2kx_2$

This pair of simultaneous can be written as a matrix equation $\quad \ddot{\boldsymbol{x}} = -\boldsymbol{A}\boldsymbol{x}$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = -\frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{-------(1)}$$

This is a second-order linear differential equation of the form,

$$\frac{d^2\boldsymbol{X}}{dt^2} + \boldsymbol{AX} = 0$$

where $\boldsymbol{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and $\boldsymbol{A} = \frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

If there is no damping in the system then the solution is purely oscillatory.

Task 1

Solving the differential equation

If we assume that the solutions of the second-order differential equation are periodic, as the bodies are oscillating, then we can let $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \cos(\omega t + \theta)$

where are α_1 and α_2 are constants and ω is a constant related to the frequency of oscillation.

Show that $$\frac{d^2\boldsymbol{X}}{dt^2} = -\omega^2\boldsymbol{X}$$

Task 2

Solving the characteristic equation

Comparing this equation with equation 1 above we see that $\mathbf{A}\boldsymbol{X} = \omega^2\boldsymbol{X}$ -------(2)

The problem is now to find the **eigenvalues** and **eigenvectors** of the matrix $\mathbf{A}$.

$\boldsymbol{X}$ is the eigenvector of the matrix $\mathbf{A}$ with corresponding eigenvalue ω^2.

Equation 2 can be rewritten as $(\mathbf{A} - \omega^2\boldsymbol{I})\boldsymbol{X} = 0$ where $\boldsymbol{I}$ is the identity matrix.

Solve the characteristic equation to find the eigenvalues, ω^2.

Hence show that the matrix has two eigenvectors $x_1(t) = x_2(t)$ and $x_1(t) = -x_2(t)$, which represent the two normal modes of oscillation.

The frequency of oscillation, $f = \frac{\omega}{2\pi}$. Find the frequency of oscillation of each mode.

Solutions

Task 1

Let $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \cos(\omega t + \theta)$

Differentiating w.r.t. time $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = -\omega \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \sin(\omega t + \theta)$

Differentiating again w.r.t. time $\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = -\omega^2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \cos(\omega t + \theta)$

Therefore $\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = -\omega^2 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\boldsymbol{\frac{d^2X}{dt^2} = -\omega^2 X}$$

Task 2

For $(\mathbf{A} - \omega^2 \boldsymbol{I})\boldsymbol{X} = 0$

Substituting for A from equation 1 gives $\begin{pmatrix} 2\frac{k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & 2\frac{k}{m} - \omega^2 \end{pmatrix} \boldsymbol{X} = 0$

For the determinant to be zero,

then the characteristic equation is $\left(2\frac{k}{m} - \omega^2\right)^2 - \frac{k^2}{m^2} = 0$

$$2\frac{k}{m} - \omega^2 = \pm\frac{k}{m}$$

which has solutions $\boldsymbol{\omega^2 = \frac{k}{m}}$ and $\boldsymbol{\omega^2 = \frac{3k}{m}}$. These are the eigenvalues.

For $\omega^2 = \frac{k}{m}$, $\frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{k}{m} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$2x_1(t) - x_2(t) = x_1(t) \ ,$$

therefore $\boldsymbol{x_1(t) = x_2(t)}$ *Mode 1*

This mode of vibration is where the masses are moving in phase with each other, in the same direction at the same time, $x_1(t) = x_2(t)$ Thus inner spring has no effect and remains unstretched.

For $\omega^2 = \frac{3k}{m}$, $\frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{3k}{m}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$2x_1(t) - x_2(t) = 3x_1(t) \ ,$$

therefore $\boldsymbol{x_1(t) = -x_2(t)}$ *Mode 2*

This mode of vibration is where the masses are moving in opposite directions, with equal but opposite displacements $x_1(t) = -x_2(t)$. In this mode all the springs have an effect on the motion and so the forces on the masses are higher. This results in greater acceleration and a higher frequency of oscillation.

The motion of the bodies will depend on their initial displacement. However the eigenvectors indicate that there are two normal modes of vibration.

The frequency of oscillation in both modes is given by $f = \frac{\omega}{2\pi}$

Mode 1, the frequency $f = \frac{1}{2\pi}\frac{\sqrt{k}}{\sqrt{m}}$ and for mode 2, $f = \frac{1}{2\pi}\frac{\sqrt{3k}}{\sqrt{m}}$.

The frequency for mode 2 is $\sqrt{3}$ times that of mode 1

The beauty of this approach is that it can be extended to larger systems of 3 or more coupled oscillators. The problem of finding the normal modes remains an eigenvalue problem but involving an *n x n* matrix. The resulting differential equation would require more advanced matrix methods to solve.