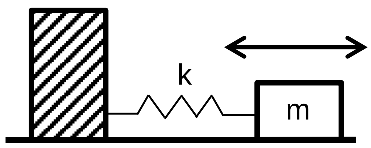
Application of A level Mathematics and Further Mathematics

This application makes use of the following:

Topics from A level Mathematics	-	Trigonometric functions and identities
	-	Integration of trigonometric functions
Topics from A level Further Mathematics	-	Second-order differential equations

Energy of an oscillating body

The motion of a body performing ***simple harmonic motion*** can be described using sine or cosine functions. Its potential and kinetic energy at any point can also be expressed using trigonometric functions.



Integration is used to find the average kinetic and potential energy.

The problem:

Consider an object of mass 4 kg sitting on a frictionless surface attached at one end of a spring. The other end of the spring is attached to a wall.

Assumptions:

Assume that the object is constrained to move horizontally.

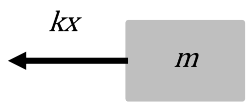
The spring has spring constant k = 200Nm^{-1}.

The tension in the spring is given by Hooke's law, $T = kx$

The spring is initially stretched a distance 2 cm from the equilibrium position and released.

Setting up the equations:

Let the displacement at time t of the object be x.



Using Newton's 2nd law gives $m\ddot{x} = -kx$ (where $\ddot{x}$ = acceleration)

$4\ddot{x} = -200x$ therefore $\ddot{x} = -50x$

The general equation for a body performing simple harmonic motion is

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{or} \quad \ddot{x} = -\omega^2 x$$

Task 1

Solving the second-order differential equation

If the displacement is given by the function $x(t) = A\cos\omega t + B\sin\omega t$.

Show by differentiation that $\frac{d^2x}{dt^2} = -\omega^2 x$.

Hence write down the general solution for $\ddot{x} = -50x$.

note 50 = ω^2

The values of constants A and B can be found by using the initial conditions when $t = 0$.

If initially the displacement $x = 0.02$ and velocity $\dot{x} = 0$ find A and B, write down the ***particular solution*** for the displacement x of the object at time t.

Task 2

Calculating the potential energy and kinetic energy

The potential energy due to the tension in a spring is given by *PE* = $\frac{1}{2}kx^2$, x is the extension of the spring and k is the spring constant, k = 200Nm^{-1}.

The kinetic energy of the object is given by *KE* = $\frac{1}{2}mv^2$, v is the velocity of the object and m is its mass, m = 4kg.

Using the particular solution from Task 1 find expressions for the potential energy and kinetic in terms of t.

Show that the total energy, E is constant.

Task 3

Calculating the average potential energy and kinetic energy

To find the average kinetic and potential energy for one oscillation we integrate the expressions for energy over an interval of one period and then divide by the time period.

The average potential energy for one oscillation is given by $\frac{1}{T}\int_0^T \frac{1}{2}kx^2.dt$

and the average kinetic energy for one oscillation is given by $\frac{1}{T}\int_0^T \frac{1}{2}kv^2.dt$

where *T* is the time period of one oscillation.

Find the period of oscillation, *T* and hence the average kinetic and potential energies.

Solutions

Task 1

$x(t) = A\cos\omega t + B\sin\omega t$ -----(1)

Differentiating w.r.t. time $\dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$ -----(2)

Differentiating w.r.t. time again $\ddot{x} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

$\ddot{x} = -\omega^2\,(A\cos\omega t + B\sin\omega t) = -\omega^2 x$

Therefore for the example $\omega^2 = 50$

The initial conditions are $t = 0$, $x = 0.02$ and $\dot{x} = 0$

Substituting into equation (1) gives 0.02 = A + 0, hence A = 0.02

Substituting into equation (2) gives 0 = 0 + B, hence B = 0

Hence the displacement at time t of the object is $\boldsymbol{x(t) = 0.02\cos\sqrt{50}t}$

Task 2

Potential energy due to the tension $= \frac{1}{2}kx^2 = 100\,(0.02\cos\sqrt{50}t)^2 = \boldsymbol{0.04\cos^2\sqrt{50}t}$ J.

Velocity of the object is $\dot{x} = -0.02\sqrt{50}\sin\sqrt{50}t$

Kinetic energy $= \frac{1}{2}m\dot{x}^2 = 2\,(0.02\sqrt{50}\sin\sqrt{50}t)^2 = \boldsymbol{0.04\sin^2\sqrt{50}t}$ J.

Total energy E $= 0.04\cos^2\sqrt{50}t + 0.04\sin^2\sqrt{50}t = 0.04\,(\cos^2\sqrt{50}t + \sin^2\sqrt{50}t)$

Therefore **E = $\boldsymbol{0.04}$** J. Hence the total energy is constant.

Task 3

One period of oscillation $T = \frac{2\pi}{w} = \boldsymbol{\frac{2\pi}{\sqrt{50}}}$.

Average potential energy $= \frac{1}{T}\int_0^T 0.04\cos^2\sqrt{50}t\ dt$

Average kinetic energy $= \frac{1}{T}\int_0^T 0.04\sin^2\sqrt{50}t\ dt$

Using the substitution $z = \sqrt{50}t$ then $dz = \sqrt{50}dt$ and the limits become 0 and 2π.

Hence the integrals are now average potential energy $= \frac{1}{2\pi}\int_0^{2\pi} 0.04\cos^2 z\ dz$

average kinetic energy $= \frac{1}{2\pi}\int_0^{2\pi} 0.04\sin^2 z\ dz$

Using these trig. identities $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$ and $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ in the integrals gives

Average P.E. $= \frac{1}{2\pi}\int_0^{2\pi} 0.04 \times \frac{1}{2}(1 + \cos 2z)\ dz$

$$= \frac{0.04}{2\pi}\int_0^{2\pi} \frac{1}{2}\ dz + \frac{0.04}{2\pi}\int_0^{2\pi} \cos 2z\ dz$$

$$= \frac{0.04}{2\pi}\left[\frac{z}{2}\right]_0^{2\pi} + \frac{0.04}{2\pi}\left[\frac{\sin 2z}{2}\right]_0^{2\pi}$$

$$= \frac{0.04}{2} + 0 = \mathbf{0.02}\text{ J}$$

Similarly you can show that the average kinetic energy is also 0.02 J.

[Note that the definite integrals $\int_0^{2\pi} \cos(nz)\ dz = \int_0^{2\pi} \sin(nz)\ dz = 0$ for any integer n.]

Interpretation

Both the kinetic and potential energy oscillate so that one is zero when the other is a maximum. The total energy changes back and forth between kinetic and potential. The sum of kinetic and potential energy is constant and on average there is an equal amount of both in every cycle of the oscillation.