GCSE Problem Solving

GCSE Problem Bank: answers and worked solutions

Let Maths take you Further…

PRC v2

30/01/14

Always Fives

If you let the original even number be $2n$ where $n = 1,2,3,4,5,6,7, \ldots$

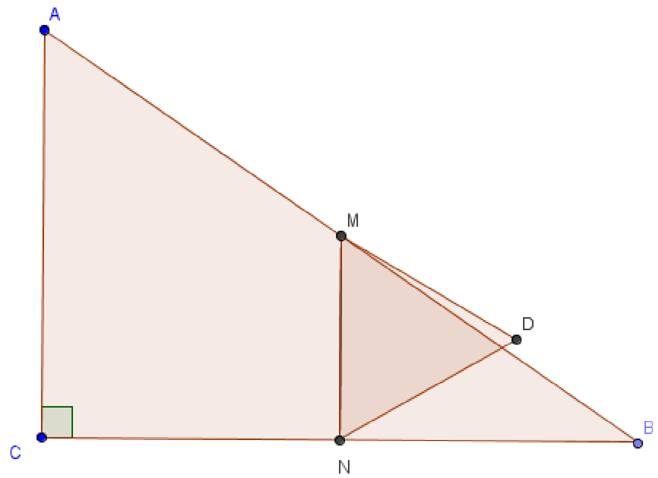
Then multiplying it by 3 gives $3 \times 2n = 6n$

Half of the original number is $\frac{1}{2} \times 2n = n$

Subtracting this gives $6n - n = 5n$

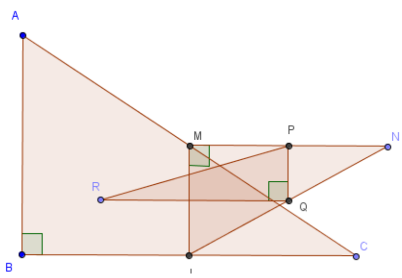
Which is always a multiple of 5 if $n = 1,2,3,4,5,6,7, \ldots$

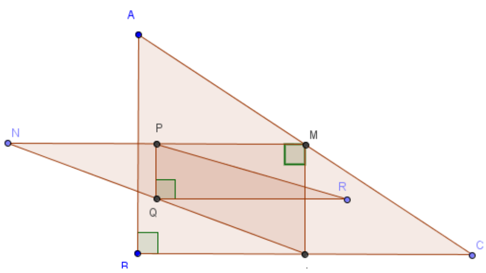
Can you draw it? – Description 1

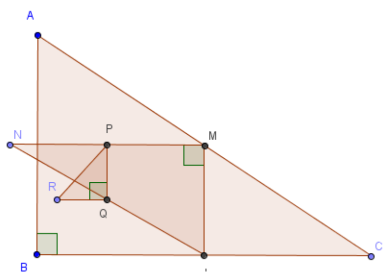


Can you draw it? – Description 2

It is important for students to realise that there is often more than one correct solution to a problem. They often see this with quadratic equations but rarely see it with geometrical problems. There are three ‘obvious’ correct sketches for this description:







Circly

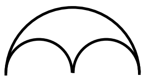
If the radius of the large circle is r, then the radius of one of the small circles is $\frac{r}{2}$

The area of the large circle is πr^2

The area of one small circle is $\pi\left(\frac{r}{2}\right)^2 = \frac{\pi r^2}{4}$

The area of both small circles is therefore $2 \times \frac{\pi r^2}{4} = \frac{\pi r^2}{2}$

This is half of the area of the large circle.

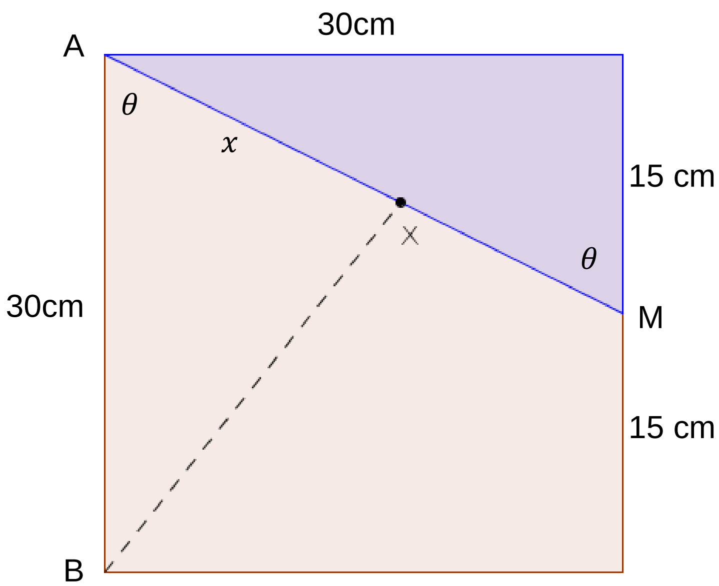
One interesting upshot of this is that the area of each  shape is equal to the area of one of the small circles.

Cube Slice

This is a challenging task

There are a number of possible approaches. This is just one of them.

The trick is to realise that it is only necessary to divide the area of the top surface into two equal sections.



Area of trapezium $= (30 \times 30) - \left(\frac{1}{2} \times 30 \times 15\right) = 900 - 225 = 675\ cm^2$

Area of triangle ABX $= 675 \div 2 = 337.5$

Area of a triangle $= \frac{1}{2}ab\sin C$

The right angled triangle that was initially removed can be used to find $\sin\theta$

hypotenuse= $\sqrt{15^2+30^2} = 15\sqrt{5}$

so $\sin\theta = \frac{30}{15\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

Using the formula $\frac{1}{2}ab\sin C$, the area of triangle ABX could also be written as $\frac{1}{2}\times 30\times x\times\sin\theta$

which simplifies to $15x\times\frac{2\sqrt{5}}{5} = 6\sqrt{5}\,x$

so $6\sqrt{5}\,x = 337.5$ giving $x = \frac{45\sqrt{5}}{4}\ cm$ or $x = 25.2\ cm$ to 3 s.f.

So the cut should go from corner B to a point that is 25.2 cm along the line AM

5 and 7 difference

This is another question with more than one possible answer

Let the numbers be a, b, c and d.

So $a+b+c+d=100$

$a-b=7$

$c-d=5$

a and b can't be the same numbers since they have a difference of 7

c and d can't be the same numbers since they have a difference of 5

So the possibilities are that $a=c$ or $a=d$ or $b=c$ or $b=d$

1) If $a=c$

$a-b=7 \Rightarrow b=a-7$

$c-d=5 \Rightarrow a-d=5 \Rightarrow d=a-5$

So $a+b+c+d=100$ can be written $a+a-7+a+a-5=100$

$4a-12=100$

$4a=112$

$a = 28$
So $b = 21$, $c = 28$ and $d = 23$

Giving the four numbers as 21, 23, 28 and 28

2) If $a = d$

$a - b = 7 \Rightarrow b = a - 7$

$c - d = 5 \Rightarrow c - a = 5 \Rightarrow d = a + 5$

So $a + b + c + d = 100$ can be written $a + a - 7 + a + a + 5 = 100$
$4a - 2 = 100$
$4a = 102$
$a = 25.5$
This is not possible since you are told that the numbers are whole numbers.

3) If $b = c$

$a - b = 7 \Rightarrow a = b + 7$

$c - d = 5 \Rightarrow b - d = 5 \Rightarrow d = b - 5$

So $a + b + c + d = 100$ can be written $b + 7 + b + b + b - 5 = 100$
$4b + 2 = 100$
$4b = 98$
$b = 24.5$
This is not possible since you are told that the numbers are whole numbers.

4) If $b = d$

$a - b = 7 \Rightarrow a = b + 7$

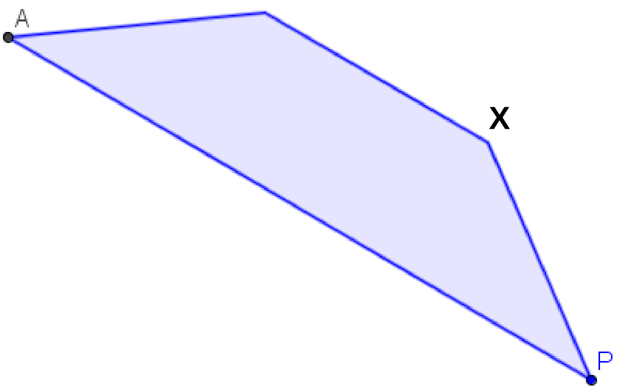
$c - d = 5 \Rightarrow c - b = 5 \Rightarrow c = b + 5$

So $a + b + c + d = 100$ can be written $b + 7 + b + b + 5 + b = 100$
$4b + 12 = 100$
$4b = 88$
$b = 22$
So $a = 29$, $c = 27$ and $d = 22$

Giving the four numbers as 22, 22, 27 and 29

Dodecagon and Hexagon

First consider this shape in the dodecagon

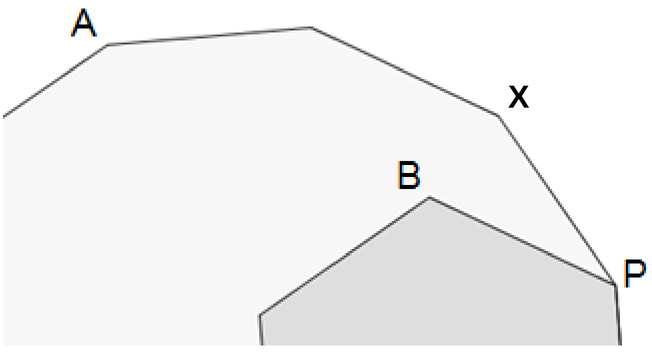


It has reflection symmetry so the angle at A is the same as the angle at P.

The internal angles of a regular dodecagon are $180 - \frac{360}{12} = 150°$

The shape is a quadrilateral so the angle XPA is $\frac{360-150-150}{2} = 30°$

Now consider the regular hexagon



The internal angle of a hexagon is $180 - \frac{360}{6} = 120°$

The angle XPB is therefore $150 - 120 = 30°$

This is the same as angle XPA so A, B and P lie on the same straight line.

Exchange Rates (PISA)

1. 1 SGD = 4.2 ZAR

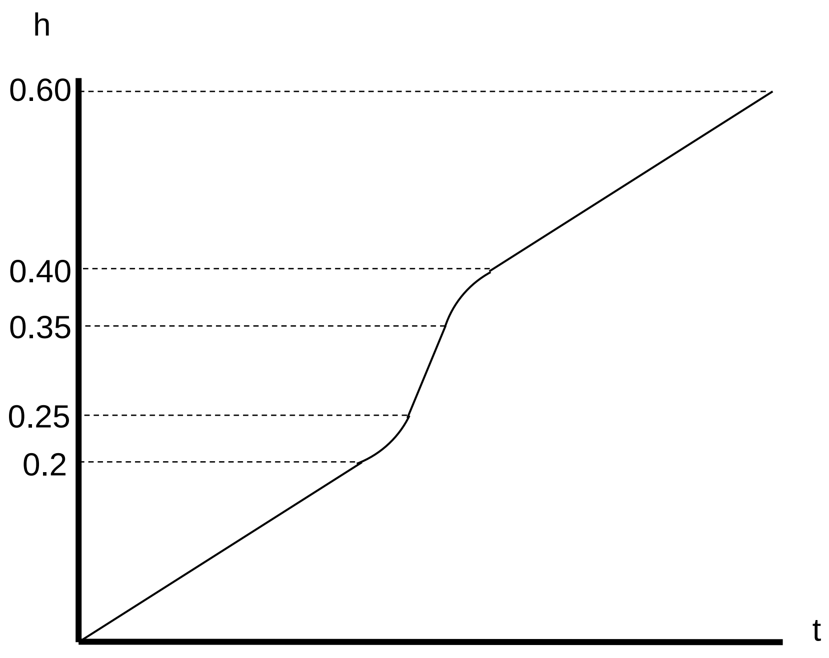
   3000 SGD = 3000 x 4.2 = 12600 ZAR

2. 1 SGD = 4.0 ZAR

   3900 ZAR = 3900 ÷ 4.0 = 975 SGD

3. If the exchange rate hadn't changed and stayed at 1 SGD = 4.2 ZAR, Mei-Ling would have received 3900 ÷ 4.2 = 928.57 SGD so the change in exchange rates was in Mei-Ling's favour

Fill it up 1

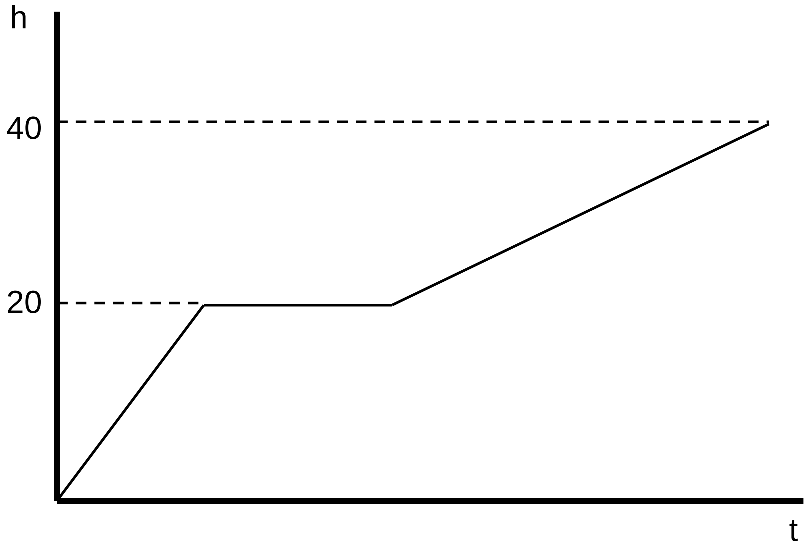


Fill it up 2

This problem merits some discussion.

Why has the phrase “the maximum height of the fuel surface” been used?

How will this change when the fuel reaches the “step” in the tank?



Growing Up (PISA)

The average height of a 20-year-old female in 1980 was 168.3 cm

The graph shows that on average the growth rate for girls slows down after 12 years of age by having a gradient that gradually decreases i.e. the graph gets less steep with time.

Females are taller than males of the same age from 11years to 13 years old

Hexabubble

The diameters of three circles added together is $27\ \mathrm{cm}$

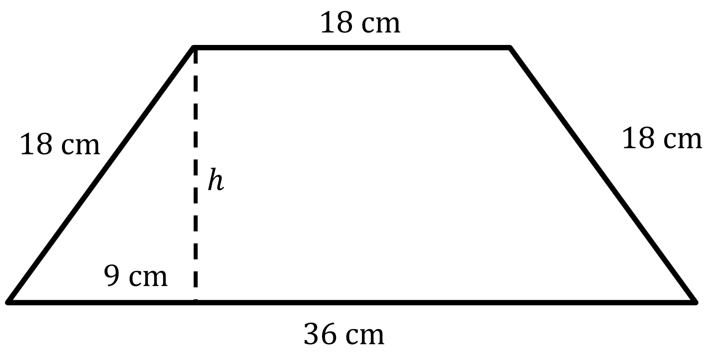
So one circle has a diameter of $9\ \mathrm{cm}$

The longer of the parallel sides is made up of three diameters and two radii (or 4 diameters) and so has length $4 \times 9 = 36\ \mathrm{cm}$

The shorter of the parallel sides is made up of one diameter and two radii (or 2 diameters) and so has length $2 \times 9 = 18\ \mathrm{cm}$

The ‘slanted’ sides of the trapezium are each made up of one diameter and two radii (or 2 diameters) and so has length $2 \times 9 = 18\text{ cm}$

The perpendicular height has to be calculated using Pythagoras’s rule



$h^2 = 18^2 - 9^2$

$h = \sqrt{18^2 - 9^2}$

$h = 9\sqrt{3}\ cm$

The area of the trapezium is therefore $= \frac{9\sqrt{3}}{2}(18 + 36) = 243\sqrt{3}\text{ cm}^2$ or 421 cm^2 (to 3 s.f.)

Line Sketch 1

This needs some thought about the line with equation $6x - 3y = 4$

Rearranging this gives $3y = 6x - 4$ which in $y = mx + c$ form is $y = 2x - \frac{4}{3}$

Sachin has the correct y intercept but his line should be parallel to the other one.

At the moment it looks like the lines could cross.

Line Sketch 2

For the line going through $(4,10)$ and $(8,12)$:

gradient $= \frac{12-10}{8-4} = \frac{1}{2}$

Using $y = mx + c$ gives initially $y = \frac{1}{2}x + c$

Using $x = 4$ and $y = 10$ as it passes through $(4,10)$ gives $10 = \frac{1}{2} \times 4 + c$ so $c = 10 - 2 = 8$

This line has equation $y = \frac{1}{2}x + 8$

For the line going through $(4,3)$ and $(12,9)$:

gradient $= \frac{9-3}{12-4} = \frac{6}{8} = \frac{3}{4}$

Using $y = mx + c$ gives initially $y = \frac{3}{4}x + c$

Using $x = 4$ and $y = 3$ as it passes through $(4,3)$ gives $3 = \frac{3}{4} \times 4 + c$ so $c = 3 - 3 = 0$

This line has equation $y = \frac{3}{4}x$

The lines cross where $\frac{3}{4}x = \frac{1}{2}x + 8$

Giving $\frac{1}{4}x = 8$

So $x = 32$ and $y = \frac{3}{4} \times 32 = 24$

The lines cross at $(32,24)$

Mystery Number

This gives the equation $7x = x + 36$

Giving $6x = 36$ so $x = 6$

Mystery Quadratic

The graph passes through $(0,0)$, $(4,0)$ and $(5,5)$.

It also appears to have a minimum of something like $(2, -4)$

Using $(0,0)$ and $(4,0)$ would suggest the equation is $y = x(x - 4)$

Testing this with $(5,5)$ gives $5 = 5(5 - 4)$ which is consistent

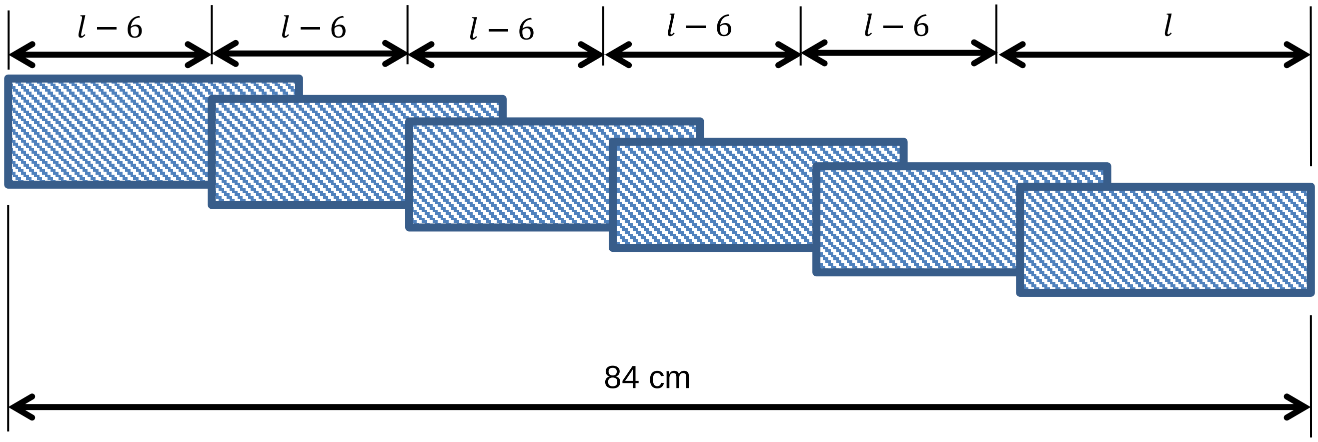
Testing with $(2, -4)$ gives $-4 = 2(2 - 4)$ which is also consistent

The equation is therefore $y = x(x - 4)$ or $y = x^2 - 4x$

Overlaps

Let the length of one tile be l cm

The uncovered length of the "bottom" tile is therefore $l - 6$ cm



The pile can be seen to consist of five lengths of $l - 6$ and one of l

So $5(l - 6) + l = 84$

$5l - 30 + l = 84$

$6l = 114$

$l = 19$ cm

How long is a piece of string?

Let the length of the shortest piece be l cm

The longest length is therefore $2l$ cm

The middle piece is therefore $l + 7$ cm

So $l + l + 7 + 2l = 79$

$4l + 7 = 79$

$4l = 72$

$l = 18$

$l + 7 = 25$

$2l = 36$

The three lengths are $18\ \mathrm{cm}$, $25\ \mathrm{cm}$ and $36\ \mathrm{cm}$

Postal Charges (PISA)

Graph C

President (PISA)

Newspaper 3

It is the most recent poll that uses a random sample.

It is better than two because it samples a larger number.

It is better than four because people that phone in are more likely to have a strong opinion which will introduce some bias.

There could be some detailed discussion following this question.

Reaction Time (PISA)

The silver medal time was 9.99 s

The bronze medal time was 10.04 s

The bronze medal reaction time was 0.216 s

Method 1

If the reaction was reduced to the minimum possible reaction time of 0.110 s, the bronze medal winner would have taken 0.106 s less time

10.04 – 0.106 – 9.934 s

which would have beaten the silver medal time

Method 2

Difference between the two times 10.04 – 9.99 = 0.05 s

Improved reaction time.by this = 0.216 – 0.05 = 0.166 s

The reaction time would have to be less than 0.166 s which is possible as this is > 0.110 s

Scaled

Five equal sections between 64 and 144

Each graduation represents $\frac{144-64}{5} = 16$

A is at $64 + 16 + 16 = 96$

B is at $96 + 16 = 112$

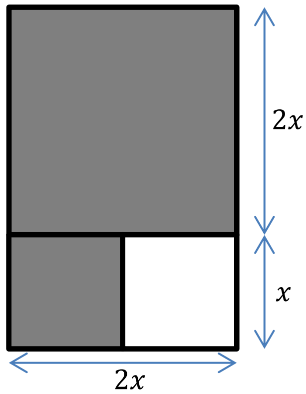
C is at $144 + 16 = 160$

Shaded squares 1

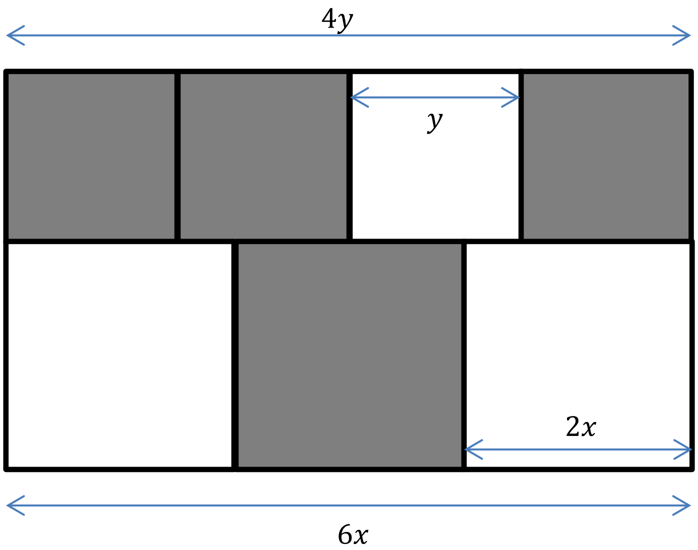
There are three sizes of square in the design.

Let the length of one side of the smallest type of square be x

The length of one side of the largest type of square is therefore $2x$

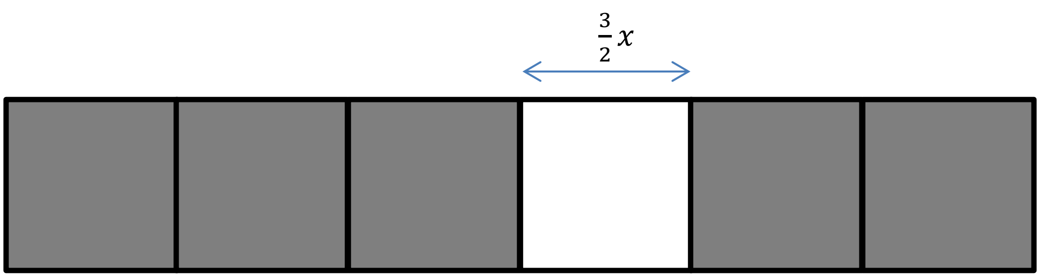


Let the length of one side of the middle type of square be y



So $4y = 6x \Rightarrow y = \frac{3}{2}x$

The length of one side of the middle type of square is $\frac{3}{2}x$



The length of one side of the overall design is therefore $= 6 \times \frac{3}{2}x = 9x$

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Area of overall design $= 9x \times 9x = 81x^2$

Area of a small square $= x^2$

Area of a middle square $= \frac{3}{2}x \times \frac{3}{2}x = \frac{9}{4}x^2$

Area of a large square $= 2x \times 2x = 4x^2$

Shaded area:

Two small squares $= 2x^2$

Four large squares $= 4 \times 4x^2 = 16x^2$

Fifteen middle squares $= 15 \times \frac{9}{4}x^2 = \frac{135}{4}x^2$

Total shaded area $= 2x^2 + 16x^2 + \frac{135}{4}x^2 = 51.75x^2$

Percentage of design shaded $= \frac{51.75x^2}{81x^2} \times 100 = 63.9\%$ (3 s.f.)

Shaded squares 2

Let the length of one side of the smallest square be x

The length of one side of the next size square is $2x$

The length of one side of the next size square is $3x$

The length of one side of the next size square is $5x$

The length of one side of the next size square is $8x$

The length of one side of the next size square is $13x$

The length of one side of the largest square is $21x$

The rectangle design measures $21x \times 34x$

The area of the overall design is $21x \times 34x = 714x^2$ square units

Shaded areas:

One each of

x^2, $(3x)^2$, $(8x)^2$ and $(21x)^2$

Total shaded area $= x^2 + 9x^2 + 64x^2 + 441x^2 = 515x^2$

Percentage of design shaded $= \frac{515x^2}{714x^2} \times 100 = 72.1\%$ (3 s.f.)

Tornado alley

This is designed to promote some discussion about the use probability in news articles.

Statement B is correct.

Each of the other statements should be discussed.

Triangle trip 1

The length of one side of each small triangle is 80 ÷ 5 = 16 cm

The route from A to B uses 8 of these lengths.

The route from A to B is 8 x 16 = 128 cm long

Triangle trip 2

The longest route is 20 x 16 = 320 cm long

Unfair Shares

If John's share is $£x$, then Sandy's is $£\frac{3}{5}x$.

So $x + \frac{3}{5}x = 640$

$\frac{8}{5}x = 640$

$8x = 3200$

$x = 400$

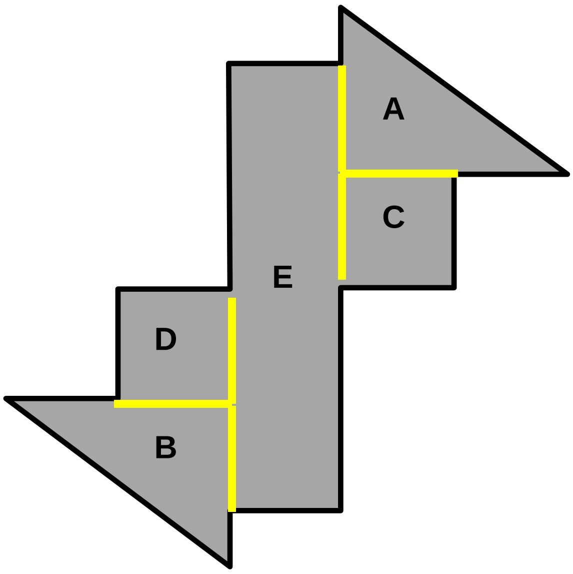
John receives $£400$

Sandy receives $\frac{3}{5} \times 400 = £240$

Wire Bending

The length of one side of a triangle is 75 ÷ 15 = 5 cm

Zag Area 1



The shape can be cut up like this:

The number of grid squares that make up each shape

can then be counted/calculated.

Triangles A and B can be put together to make

one rectangle made up of $3 \times 4 = 12$ squares

C and D are each made up of $2 \times 2 = 4$ squares

E is made up of $2 \times 8 = 16$ squares

The overall shape covers $12 + 4 + 4 + 16 = 36$ squares

If the length of one side of a grid square is $x\ m$, it's area is $x^2\ \mathrm{m}^2$

So $36x^2 = 1.44$

Giving $x^2 = 0.04$

So $x = 0.2\ \mathrm{m}$

Each grid square has a side length of $0.2\ \mathrm{m}$.

The perimeter without the two 'angled' lines is $26 \times 0.2 = 5.2\ \mathrm{m}$ (by counting square edges)

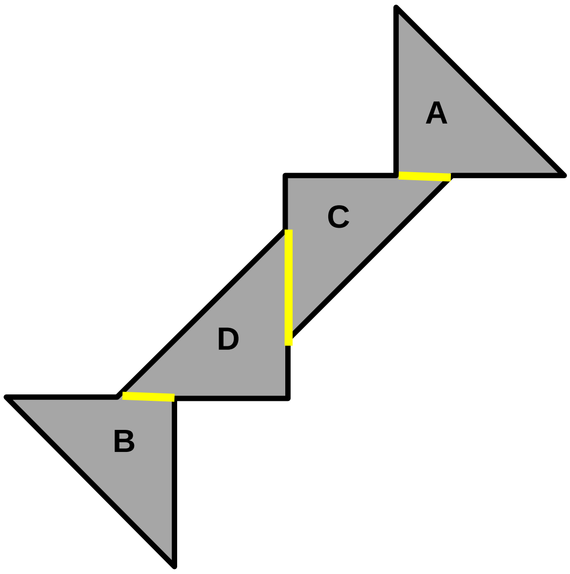
The angled sides can be calculated using Pythagoras

Length $= \sqrt{0.6^2 + 0.8^2} = 1$

The perimeter is therefore $5.2 + 1 + 1 = 7.2\ \mathrm{m}$

Zag Area 2

The shape can be cut up like this:



The number of grid squares that make up each shape can then be counted/calculated.

Triangles A and B can be put together to make one square made up of $3 \times 3 = 9$ squares

Triangles C and D can also be put together to make one square made up of $3 \times 3 = 9$ squares

The overall shape covers $9 + 9 = 18$ squares

If the length of one side of a grid square is x cm, it's area is x^2 cm^2

So $18x^2 = 450$

Giving $x^2 = 25$

So $x = 5$ cm

Each grid square has a side length of 5 cm.

The perimeter without the two 'angled' lines is $16 \times 5 = 80$ cm (by counting square edges)

The angled sides can be calculated using Pythagoras

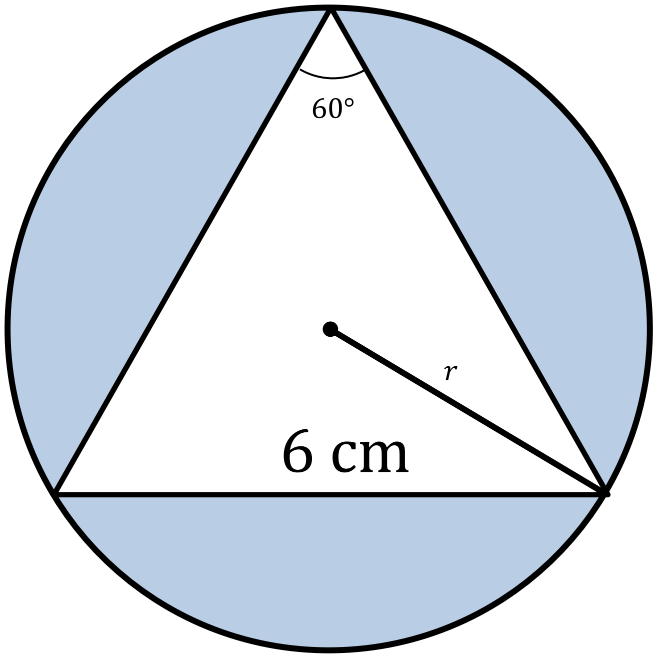
Length $= \sqrt{15^2 + 15^2} = 15\sqrt{2}$

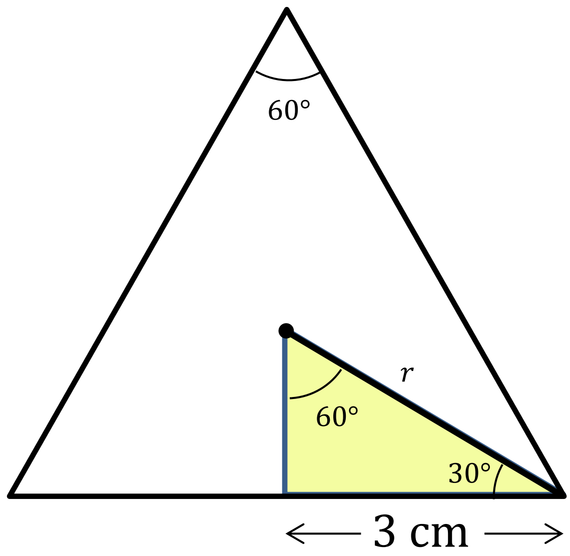
The perimeter is therefore $80 + 4 \times 15\sqrt{2} = 80 + 60\sqrt{2}$ cm or 165 cm (3 s.f.)

Circle area 1 Solution

The triangle has internal angles of $60°$

The radius of the circle is required so marking the centre of the circle and one radius (to one of the corners of the triangle gives





There are a number of ways of proving that the angles in the marked triangle are $30°$ and $60°$.

From this $r = \frac{3}{\cos 30}$ so $r = 2\sqrt{3}$ cm

The other side of the triangle is $\sqrt{3}$ cm.

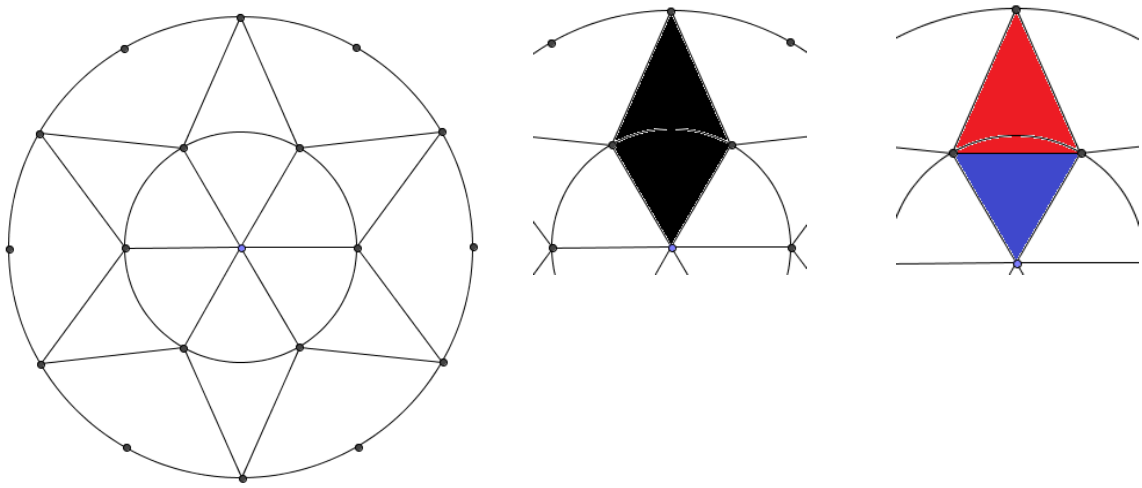
The area of the equilateral triangle can be found by using six of the shaded right angle triangles (there are plenty of other ways).

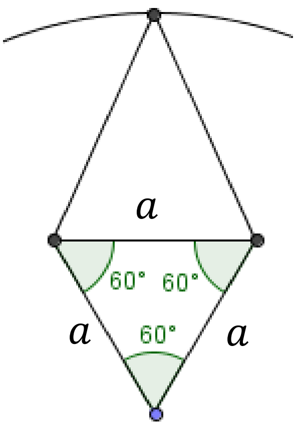
Area of equilateral triangle $= 6 \times \frac{1}{2} \times 3 \times \sqrt{3} = 9\sqrt{3}\ \mathrm{cm}^2$

Area of circle $= \pi \times \left(2\sqrt{3}\right)^2 = 12\pi\ \mathrm{cm}^2$

So the shaded area $= 12\pi - 9\sqrt{3}\ \mathrm{cm}^2$. This is $22.1\ \mathrm{cm}^2$ to 1 d.p.

Shaded circles 1 Answer





$$2a - a\sin 60$$

$$= \frac{(4-\sqrt{3})a}{2}$$

$$a\sin 60$$

$$= \frac{\sqrt{3}}{2}a$$

Area of small triangle

$= \frac{1}{2} \times a \times \frac{\sqrt{3}}{2}a$

$= \frac{\sqrt{3}}{4}a^2$

Area of larger triangle

$= \frac{1}{2} \times a \times \frac{(4-\sqrt{3})}{2}a$

$= \frac{(4-\sqrt{3})}{4}a^2$

Area of kite

$= \frac{\sqrt{3}}{4}a^2 + \frac{(4-\sqrt{3})}{4}a^2$

$= a^2$

Area of 6 kites $= 6a^2$

Area of small circle $= \pi a^2$

Shaded area $= 6a^2 - \pi a^2 = a^2(6-\pi)$

Area of large circle $= 4\pi a^2$

Shaded proportion $= \frac{6-\pi}{4\pi}$ as a fraction. This is 22.7%

PRC v2

30/01/14

Is Greenland bigger than Africa 1?

Depending on how the scale is used, answers in the order of

7,875,000 km^2 for Greenland

and

7,750,000 km^2 for Africa

Making it look like Greenland is bigger than Africa!

Is Greenland bigger than Africa 2?

The Peter's projection gives a much fairer account of area.

Depending on how the scale is used, answers in the order of

4,250,000 km^2 for Greenland

and

30,222,000 km^2 for Africa

These answers are actually still a little on the high side for both. It is worth looking up the actual values from a web site.

There is quite a lot of discussion that can be had from both of these results, both socio-political and mathematical.