Application of A level Mathematics and Further Mathematics

This application makes use of the following:

Topics from AS level Further Mathematics

- Forming a 2x2 matrix from a pair of simultaneous equations
- Finding the inverse of a 2x2 matrix.

Mesh Currents

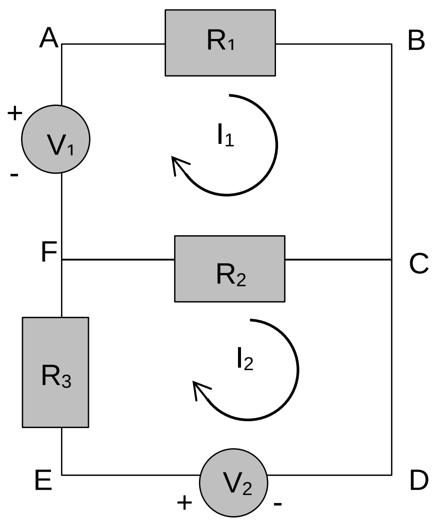
An electrical network can be modelled by a set of simultaneous equations which can be represented and solved efficiently using matrix methods.

A ***mesh*** loop in an electrical network is a loop that does not contain any smaller loops.

The diagram below shows two meshes: ABCF and CDEF.

The mesh currents I_1 and I_2 are shown as running clockwise. The net current in a branch of the network is called the ***branch current*** and these are found by combining the mesh currents.

For example the branch current from C to F is $I_1 - I_2$.



The problem: Calculate the mesh currents, I_1 and I_2.

Setting up the model:

Kirchhoff's voltage law states that the sum of voltages around a mesh is zero. This means that the voltage rises are equal to the voltage drops across a resistor, where V = IR.

Using Kirchhoff's Law for each mesh loop you can form two equations in V, I and R.

Equation 1 for mesh ABCF: $V_1 = I_1 R_1 + (I_1 - I_2)R_2 \quad = I_1 (R_1 + R_2) - I_2 R_2$

Equation 2 for mesh CDEF: $V_2 = I_2 R_3 + (I_2 - I_1)R_2 \quad = - I_1 R_2 + I_2 (R_2 + R_3)$

The information can be written as a matrix equation of the form $\mathbf{R}\boldsymbol{I} = \boldsymbol{V}$, where $\boldsymbol{R}$ is a 2x2 matrix, and $\boldsymbol{I}$ and $\boldsymbol{V}$ are vectors.

$$\begin{pmatrix} (R_1 + R_2) & -R_2 \\ -R_2 & (R_2 + R_3) \end{pmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Solving this matrix equation involves finding the inverse of the 2x2 matrix.

Task 1

For a particular electric network the voltages $V_1 = 5V$ and $V_2 = 10V$ and the resistances $R_1 = 3\Omega$, $R_2 = 8\ \Omega$, and $R_3 = 6\ \Omega$.

Substitute these values into the matrix equation and simplify.

Task 2

Solve the matrix equation by finding the inverse of matrix $\boldsymbol{R}$.

Show that the mesh currents I_1 and I_2 are equal.

Hence find the branch current between C and F.

Solutions

Task 1

Using values given $\boldsymbol{R} = \begin{pmatrix} 11 & -8 \\ -8 & 14 \end{pmatrix}$

Therefore $\begin{pmatrix} 11 & -8 \\ -8 & 14 \end{pmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

Task 2

$$\boldsymbol{R}^{-\mathbf{1}} = \frac{\mathbf{1}}{\mathbf{90}} \begin{pmatrix} 14 & 8 \\ 8 & 11 \end{pmatrix}$$

Solving the matrix equation

$$\frac{\mathbf{1}}{\mathbf{90}} \begin{pmatrix} 14 & 8 \\ 8 & 11 \end{pmatrix} \begin{pmatrix} 11 & -8 \\ -8 & 14 \end{pmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{\mathbf{1}}{\mathbf{90}} \begin{pmatrix} 14 & 8 \\ 8 & 11 \end{pmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{\mathbf{1}}{\mathbf{90}} \begin{bmatrix} 14 \times 5 + 8 \times 10 \\ 8 \times 5 + 11 \times 10 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{\mathbf{1}}{\mathbf{90}} \begin{bmatrix} 150 \\ 150 \end{bmatrix} = \frac{\mathbf{1}}{\mathbf{3}} \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Therefore $I_1 = 1.67$ A and $I_2 = 1.67$A.

Hence the branch current between C and F = $I_1 - I_2 = 0$.

This example is of a simple electrical network. More complex electrical networks can be analysed using larger matrices.