

Application of A level Mathematics and Further Mathematics

This application makes use of the following:

Topics from A level Mathematics	- Trigonometric functions
	- Transformations of graphs
	- Radian measure
Topics from AS Further Mathematics	- Complex numbers
	- 2x2 matrices
Topics from A2 Further Mathematics	- Euler's relation for complex numbers

Alternating currents in electrical circuits

An important application of complex numbers is in the study of alternating current (A.C.) circuits.

The phase relationship between the Resistor, Inductor and Capacitor can be expressed more efficiently using complex numbers.

The current and voltage an A.C. circuit can be expressed using sine functions.

$$\text{Current is } i(t) = I \sin \omega t$$

and Voltage is $v(t) = V \sin(\omega t + \alpha)$

where I and V are the peak voltage and current, ω is the angular frequency, ($f = \frac{\omega}{2\pi}$), and α is the phase angle.

For a **resistor of resistance R**, Ohm's law states that the voltage across the resistor $v_R(t) = Ri$.

The current and voltage are in phase, hence if $i(t) = I \sin \omega t$ then $v_R(t) = RI \sin \omega t$.

For an **inductor of inductance L**, Faraday's law states that the voltage across the inductor $v_L(t) = \omega Li$.

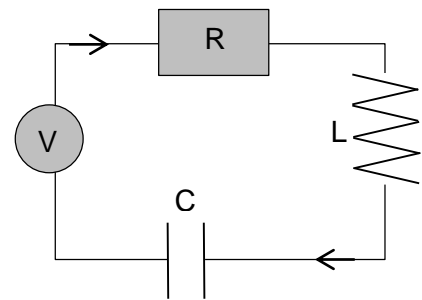
The current and voltage are not in phase, the voltage peaks before the current by $\frac{\pi}{2}$.

Hence if $i(t) = I \sin \omega t$ then $v_L(t) = \omega LI \sin(\omega t + \frac{\pi}{2})$.

For a **capacitor of capacitance C**, the voltage across the capacitor $v_C(t) = \frac{i}{\omega C}$.

The current and voltage are not in phase, the voltage peaks after the current by $\frac{\pi}{2}$.

Hence if $i(t) = I \sin \omega t$ then $v_C(t) = \frac{I}{\omega C} \sin(\omega t - \frac{\pi}{2})$.



Task 1

Sketch a graph to show how the current $i(t) = I \sin \omega t$ varies with time.

On the same axes sketch the graphs of the voltages v_R , v_L and v_C across the resistor, inductor and capacitor, respectively.

Task 2

Euler's relations are, $e^{j\theta} = \cos\theta + j\sin\theta$ and $e^{-j\theta} = \cos\theta - j\sin\theta$, where $j = \sqrt{-1}$.

Show that

$$v = \begin{cases} v_R(t) = \mathbf{Im}(RIe^{j\omega t}) & \text{for the resistor} \\ v_L(t) = \mathbf{Im}(\omega LIe^{j(\omega t + \frac{\pi}{2})}) & \text{for the inductor} \\ v_C(t) = \mathbf{Im}(\frac{I}{\omega C}e^{j(\omega t - \frac{\pi}{2})}) & \text{for the capacitor.} \end{cases}$$

where **Im** indicates the imaginary part of the complex number.

These three expressions have a common term of $Ie^{j\omega t}$.

Hence the voltage can be written as $v = \mathbf{Im}(ZIe^{j\omega t})$, where

$$Z = \begin{cases} R & \text{for the resistor} \\ \omega L e^{j\frac{\pi}{2}} & \text{for the inductor} \\ \frac{1}{\omega C} e^{-j\frac{\pi}{2}} & \text{for the capacitor.} \end{cases}$$

Z is called the **complex impedance** and $V = IZ$ is called the **complex voltage**.

The complex voltage for a circuit is the algebraic sum of the complex voltages for each individual element in the circuit.

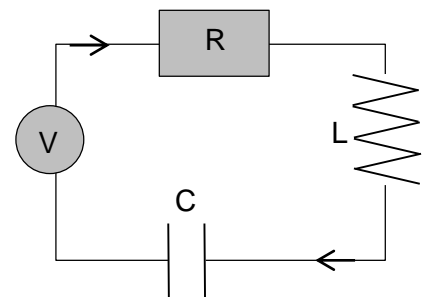
Show that $V = IR + j\omega IL - \frac{jI}{\omega C}$.

Task 3

In the circuit shown here an alternating voltage V of 100 volts is applied with a frequency of 50 hertz.

The resistance is 12 ohms, the inductance $L = 0.15$ henrys and the capacitance is 100 micro-farads (100×10^{-6} farads).

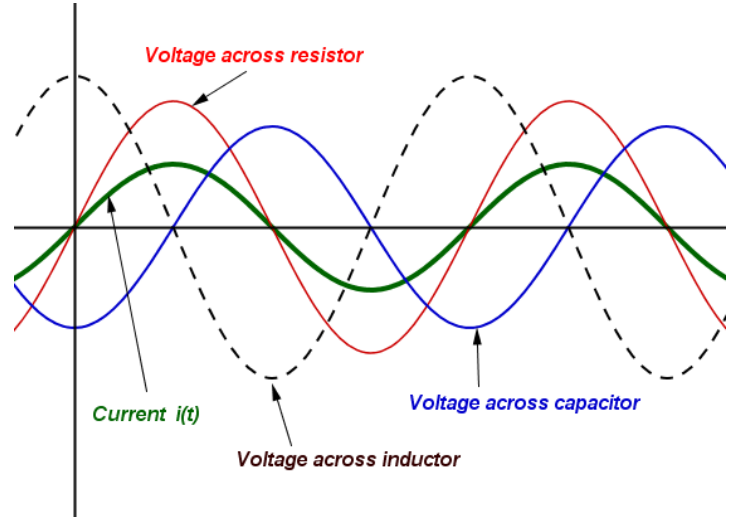
- Find the angular frequency ω .
- Find the complex impedance for this circuit.
- Using $V = IZ$, find the complex current for the circuit.



Solutions

Task 1

Sketching graphs to show how the current $i(t) = I \sin \omega t$ varies with time, together with the voltages v_R , v_L and v_C across the resistor, inductor and capacitor, respectively, we can see the phase shifts indicated by the trigonometric functions.



Task 2

Euler's relation gives $e^{j\omega t} = \cos \omega t + j \sin \omega t$, where $j = \sqrt{-1}$.

For the resistor, $v_R(t) = RI \sin \omega t = \text{Im}(RIe^{j\omega t})$

For the inductor, $v_L(t) = \omega LI \sin \left(\omega t + \frac{\pi}{2} \right) = \text{Im} \left(\omega LI e^{j(\omega t + \frac{\pi}{2})} \right)$

For the capacitor, $v_C(t) = \frac{I}{\omega C} \sin \left(\omega t + \frac{\pi}{2} \right) = \text{Im} \left(\frac{I}{\omega C} e^{j(\omega t - \frac{\pi}{2})} \right)$

Complex impedance, $Z = \begin{cases} R & \text{for a resistor} \\ \omega L e^{j\frac{\pi}{2}} & \text{for an inductor} \\ \frac{1}{\omega C} e^{-j\frac{\pi}{2}} & \text{for a capacitor.} \end{cases}$

$V = IZ$, $V = \begin{cases} IR & \text{for a resistor} \\ \omega I L e^{j\frac{\pi}{2}} = \omega I L \left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2} \right) = j \omega I L & \text{for an inductor} \\ \frac{I}{\omega C} e^{-j\frac{\pi}{2}} = \frac{I}{\omega C} \left(\cos \frac{-\pi}{2} + j \sin \frac{-\pi}{2} \right) = -\frac{jI}{\omega C} & \text{for a capacitor.} \end{cases}$

Therefore $V = IR + j\omega IL - \frac{jI}{\omega C}$

Task 3

a) The angular frequency $\omega = 2\pi f = 6.28 \times 50 = 314.2$

b) The complex impedance $Z = R + j\omega L - \frac{j}{\omega C}$

$$= 12 + j \times 314.2 \times 0.15 - \frac{j}{314.2 \times 100 \times 10^{-6}}$$

$$= 12 + 47.13j - 31.83j = \mathbf{12 + 15.3j}$$

c) Using $V = IZ$, the complex current $I = \frac{100}{12+15.3j}$

$$\text{Simplifying this gives } I = \frac{100}{12+15.3j} \times \frac{12-15.3j}{12-15.3j} = \frac{1200-1530j}{144+234}$$

The complex current $I = \mathbf{3.17 - 4.05j}$

In an A.C. circuit with multiple components the handling of multiple sine and cosine functions to represent the voltages and currents becomes difficult. The use of complex numbers in the form of exponential functions eases the difficulty. The complex voltage for a LCR circuit is the algebraic sum of the complex voltages for each individual element in the circuit.