Integrating Problem Solving into A Level Mathematics Lessons: Integrating problem solving into lessons

Aim	To develop the students' problem solving ability by integrating problems into lessons.
Method	Using questions that are part problems (i.e. a mixture of structured question and problem) and questions that allow further skills to be developed is an important part of developing students' problem solving skills. These sorts of problems will resemble what the students will see in an examination.
Activity	These problems can be presented using techniques from the earlier videos. The problems do not form a lesson on their own but illustrate the sorts of problems that can be easily integrated into lessons at key points.

Without using a calculator, show that

$$\sqrt{3}-\sqrt{2}=\sqrt{5-2\sqrt{6}}$$

Find better ways to write

(i) $\sqrt{12-2\sqrt{35}}$

(ii) $\sqrt{13+2\sqrt{42}}$

(iii) $\sqrt{21+6\sqrt{10}}$

AS/A Level

Use and manipulate surds, including rationalising the denominator

The language here is deliberately vague.

Once the students have completed the initial question they should realise that there are other ways of writing each of the following expressions that are "better" as they do not include nested square roots.

The problem here is to spot a quick way to get each answer and then explain why it works.

The second example used in the video is shown on the next page

Write down the first five terms of

$$\sum_{r=1}^{10} r^2$$

Write down the first five terms of

$$\sum_{r=1}^{10} (r-1)^2$$

Use your answers to help you find the value of

$$\sum_{r=1}^{10} r^2 - \sum_{r=1}^{10} (r-1)^2$$

Find an expression in terms of n for the value of

$$\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} (r-1)^2$$

Find an expression in terms of m and n (where $n > m$) for the value of

$$\sum_{r=m}^{n} r^2 - \sum_{r=m}^{n} (r-1)^2$$

A Level

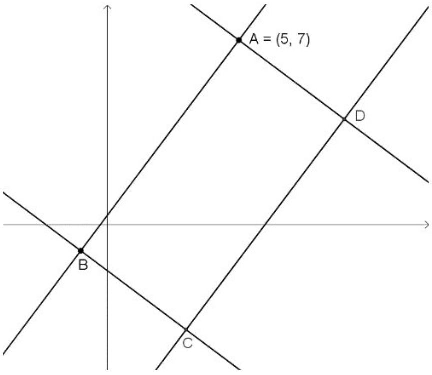
Understand and use sigma notation for sums of series

This is a problem for the students as it moves from something that they should know about (from A level mathematics) to something that is part of A level further mathematics (the method of differences).

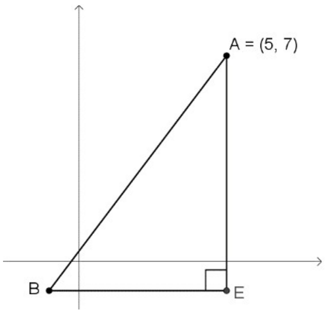
For additional ideas and transcripts of lessons in which these techniques have been applied, see the MEI's Mathematical Problem Solving: A Guide for teachers at mei.org.uk/problem-solving-guide.

Solution to the problem

Labelling the points of intersection A, B, C and D as shown:



For $AB = 10$ the right-angled triangle AEB (shown below) must be a Pythagorean triple with 10 as the hypotenuse since all of the points are at integer values. There is only one triple that has this property, a $\{6,8,10\}$ triangle.



From the fact that the line passes above the origin (i.e. the gradient of $AB > 1$) it can be seen that $AE > BE$ so from the Pythagorean triple $AE = 8$ and $BE = 6$ giving the point B at $(-1,-1)$.

Note: were the lengths swapped i.e. $AE = 6$ and $BE = 8$, the point B would be at $(-3,1)$ which it clearly isn't as it is below the x axis.

The gradient of $AB = \frac{8}{6} = \frac{4}{3}$. The gradient of $BC = -\frac{3}{4}$.

For integer coordinates we now have a $\{3,4,5\}$ triangle or one that is a multiple of $\{3,4,5\}$. Since $BC < AB$ (which the students should have established in their questioning), only a $\{3,4,5\}$ is possible.

This places point C at $(3,-4)$ and, since AD is parallel to BC and AB is parallel to CD, point D at $(9,4)$.

The equations of the lines can now be found

The line through AB: gradient $= \frac{4}{3}$, through $(-1,-1)$ $\quad y + 1 = \frac{4}{3}(x + 1)$

This simplifies to $4x - 3y + 1 = 0$.

The line through CD: gradient $= \frac{4}{3}$, through $(3,-4)$ $\quad y + 4 = \frac{4}{3}(x - 3)$

This simplifies to $4x - 3y - 24 = 0$.

The line through BC: gradient $= -\frac{3}{4}$, through $(-1, -1)$ $\quad y + 1 = -\frac{3}{4}(x + 1)$

This simplifies to $3x + 4y + 7 = 0$.

The line through AD: gradient $= -\frac{3}{4}$, through $(5,7)$ $\quad y - 7 = -\frac{3}{4}(x - 5)$

This simplifies to $3x + 4y - 43 = 0$.

Mathematical problem solving

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Mathematical problem solving

Write down the first five terms of $\sum_{r=1}^{10} r^2$
Write down the first five terms of $\sum_{r=1}^{10} (r-1)^2$
Use your answers to help you find the value of $\sum_{r=1}^{10} r^2 - \sum_{r=1}^{10} (r-1)^2$
Find an expression in terms of n for the value of $\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} (r-1)^2$
Find an expression in terms of m and n (where $n > m$) for the value of $\sum_{r=m}^{n} r^2 - \sum_{r=m}^{n} (r-1)^2$