EdExcel Further Pure 1

Complex Numbers

Section 3: Complex numbers and equations

Notes and Examples

These notes contain subsections on:

- Equating real and imaginary parts
- Roots of complex numbers
- Polynomial equations with complex roots

Equating real and imaginary parts

For two complex numbers to be equal, then the real parts must be equal and the imaginary parts must be equal. So one equation involving complex numbers can be written as two equations, one for the real parts, one for the imaginary parts.

The example below shows how this technique can be used to solve equations involving complex numbers. In Section 1 (Example 6), this equation was solved by treating it in the same sort of way as for an equation involving real numbers, using division of complex numbers. Here, the method of equating real and imaginary parts is used.

Example 1
Solve the equation

$$(3 - 2i)(z - 1 + 4i) = 7 + 4i$$

Solution
Let $z = x + iy$

$$(3 - 2i)(x + iy - 1 + 4i) = 7 + 4i$$
$$(3 - 2i)((x - 1) + i(y + 4)) = 7 + 4i$$
$$3(x - 1) - 2i(x - 1) + 3i(y + 4) - 2i^2(y + 4) = 7 + 4i$$
$$3(x - 1) - 2i(x - 1) + 3i(y + 4) + 2(y + 4) = 7 + 4i$$

Equating real parts: $3(x - 1) + 2(y + 4) = 7 \quad \Rightarrow 3x + 2y = 2$ ①
Equating imaginary parts: $-2(x - 1) + 3(y + 4) = 4 \quad \Rightarrow -2x + 3y = -10$ ②

① × 2 $\quad 6x + 4y = 4$
② × 3 $\quad -6x + 9y = -30$

Adding: $\quad 13y = -26$
$y = -2$
$x = 2$

$z = 2 - 2i$

Roots of complex numbers

The technique of equating real and imaginary parts can also be used for finding the square root of a complex number.

Example 2
Find the square root of 16 – 30i.

Solution

$(a + bi)^2 = 16 - 30i$
$a^2 + 2abi + b^2i^2 = 16 - 30i$
$a^2 + 2abi - b^2 = 16 - 30i$

Equating imaginary parts: $2ab = -30 \Rightarrow b = -\frac{15}{a}$

Equating real parts: $a^2 - b^2 = 16$

Substituting: $a^2 - \frac{225}{a^2} = 16$

Multiplying through by a^2: $a^4 - 225 = 16a^2$

$a^4 - 16a^2 - 225 = 0$

This is a quadratic in a^2 and can be factorised:

$(a^2 - 25)(a^2 + 9) = 0$

Since a is real, $a^2 + 9$ cannot be equal to zero.
Therefore $a = 5$ or $a = -5$.

$a = 5 \Rightarrow b = -\frac{15}{a} \Rightarrow b = -\frac{15}{5} = -3$

$a = -5 \Rightarrow b = -\frac{15}{a} \Rightarrow b = -\frac{15}{-5} = 3$

So the square roots of 16 – 30i are 5 – 3i and -5 + 3i.

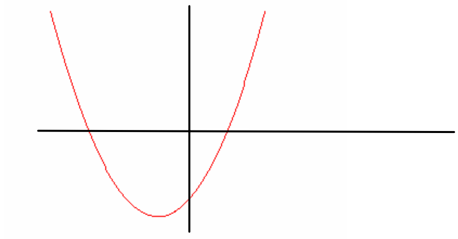
Note that as with real numbers, one square root is the negative of the other. However, it does not make sense to talk about “the positive square root” or “the negative square root”.

You can find fourth roots by finding the square root of each square root. Cube roots are usually more difficult. In Further Pure 3 a different approach to finding any root of a complex number is studied.

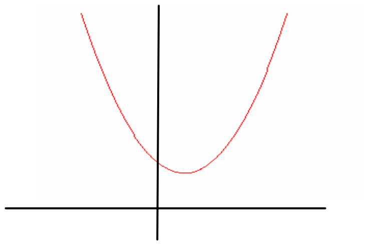
Polynomial equations with complex roots

You now know from your work on complex numbers that every quadratic equation has exactly two solutions, if you count repeated roots and complex roots.

There are two possibilities:

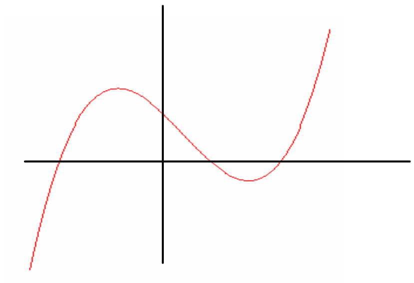


The graph crosses the x axis twice. There are two real distinct roots. If the graph just touches the axis, then the root is repeated, but this still counts as two roots.

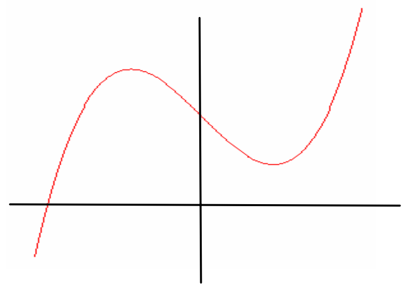


The graph does not cut the x axis. There are two complex roots which are a conjugate pair.

For a cubic equation, there are also two possibilities:



The graph cuts the x axis three times. In the diagram there are three real distinct roots. However, two of the roots could be the same, in which case the graph would touch the axis at one of the turning points, or all three roots could be the same, in which case there would be a point of inflection on the x axis.



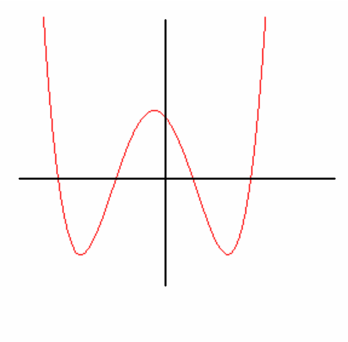
Here the graph cuts the x axis only once. There is one real root, and there is also a conjugate pair of complex roots.

You can see graphically that a cubic equation must have at least one real root. If the term in x^3 is positive, then for large positive values of x the value of the function is large and positive, and for large negative values of x the value of the function is large and negative. (If the term in x^3 is negative, this is reversed). So all cubic graphs must cut the x axis at least once.

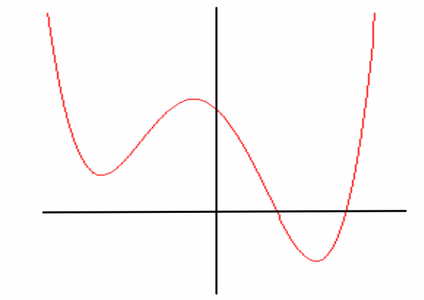
Of course, the real root may not be an integer or even a rational number, so you may not be able to find it! However, any cubic equations you meet in this section will have a simple real root, so that you can solve it.

A formula does exist for solving all cubic equations, but it is extremely complicated. Find out more **HERE** and **HERE**

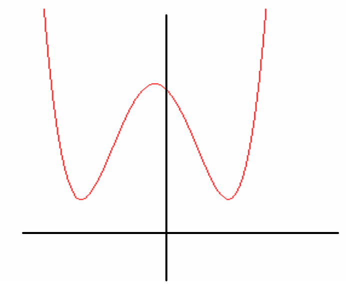
For quartic equations, there are three possibilities:



There may be four real roots, some of which may be repeated.



There may be two real roots, and a conjugate pair of complex roots.



There may be no real roots. In this case, there are two conjugate pairs of complex roots.

There is, again, an extremely complicated formula for the roots of a quartic equation. Find out more **HERE**.

For higher degree equations, there are no general formulae to find the roots. It is not simply that no-one has managed to find them yet: it was proved by **Galois** (a very interesting character) that no such formulae exist for any polynomial equations higher than quartic. If there happens to be one or more integer root which you can find by trial and error, it may be possible to solve a higher degree equation by algebraic methods. Otherwise, there are numerical methods which provide approximate solutions.

The main difficulty in proving that any polynomial equation of degree n has exactly n roots is proving that any polynomial has at least one root. If you assume that a polynomial of degree n has at least one root, then you can express the polynomial as a product of a linear factor and a polynomial of degree n-1. Then, since the assumption that any polynomial has at least one root also holds for the new polynomial of degree n-1, then you can express this polynomial as a product of a linear factor and a polynomial of degree n-2. And so on, until the polynomial has been factorised into n linear factors, giving n roots. (This applies even if the roots are irrational or complex).
For example, if you know one root of a quintic equation, you can express it as the product of a linear factor and a quartic factor. Then since a quartic equation must have at least one root, you can express the quartic factor as the product of a linear factor and a cubic factor. Since a cubic equation has at least one root, you can express the cubic factor as the product of a linear factor and a quadratic factor, which can be factorised using the quadratic formula.

So, if we can prove that all polynomial equations have one root, then we can prove that a polynomial equation of degree n has exactly n roots using the method above. (This is an example of proof by induction, in which you show that if a statement is true for n, then it is also true for n+1. The proof has been stated very informally here). Proving that all polynomial equations have at least one root is much more difficult: there are a number of approaches, all well beyond 'A' level. You can use a graphical approach to show that all polynomials of odd degree have at least one root, as described above for cubics, however, this is not a rigorous proof!

In practice, the situations you are likely to encounter include

- cubics where you are given one complex root. In this case you can deduce a second complex root which is the conjugate of the first, and use these two roots to find a quadratic factor of the cubic. You can then factorise the cubic into the quadratic factor and a linear factor (by inspection or polynomial division) and deduce the third (real) root from the linear factor.

- cubics where you are given the real root (or told that an integer root exists, which you can find by trial and error). In this case you can factorise the cubic into a linear factor and a quadratic factor, by inspection or polynomial division, and then use the quadratic formula to find the other two roots. Example 3 below shows a problem of this type.

- quartics where you are given a complex root. In this case you can again deduce a second complex root which is the conjugate of the first, and find a quadratic factor. You can then factorise the quartic into two quadratics, and use the quadratic formula to find the other two roots (which could be real or complex). Example 4 below shows a problem of this type.

- quartics where you are given one or two real roots, or told that they exists. Find the real roots by trial and error if you need to, then factorise the quartic into the two known linear factors and a quadratic factor, which you can then use to find the other two roots.

If you have already covered the work on the Factor Theorem in Core 2, you will be familiar with the techniques of factorising a polynomial by inspection or by polynomial division. If not, look here for **information** on the factor theorem.

The following two examples show the kind of examples you need to be able to do. In each case factorising is involved. There are PowerPoint presentations to show the different methods you can use for the factorising in each case.

Example 3
The equation $z^3 - z^2 - 4z - 6 = 0$ has an integer root. Find all the roots of the equation.

Solution

Let $f(x) = z^3 - z^2 - 4z - 6$

$f(1) = 1 - 1 - 4 - 6 = -10$

$f(2) = 8 - 4 - 8 - 6 = -10$

$f(3) = 27 - 9 - 12 - 6 = 0$

Therefore (z – 3) is a factor by the factor theorem.

$z^3 - z^2 - 4z - 6 = 0$

$(z-3)(z^2 + 2z + 2) = 0$

Here the cubic equation has been factorised into the known linear factor and a quadratic factor. The PowerPoint presentation **Factorising cubics** demonstrates some different methods for doing this.

The other roots are the roots of the quadratic equation $z^2 + 2z + 2 = 0$.
Using the quadratic formula:

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2} \\ &= \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= -1 \pm i \end{aligned}$$

The roots are 3, -1 + i and -1 – i.

Example 4

Show that -2 + i is one root of the quartic equation $z^4 + 2z^3 + 2z^2 + 10z + 25 = 0$, and find the other roots.

Solution

$z = -2 + i$

$z^2 = (-2+i)^2 = 4 - 4i - 1 = 3 - 4i$

$z^3 = (3-4i)(-2+i) = -6 + 11i + 4 = -2 + 11i$

$z^4 = (-2+11i)(-2+i) = 4 - 24i - 11 = -7 - 24i$

Substituting into $z^4 + 2z^3 + 2z^2 + 10z + 25$:

$$\begin{aligned} &-7 - 24i + 2(-2+11i) + 2(3-4i) + 10(-2+i) + 25 \\ &= -7 - 24i - 4 + 22i + 6 - 8i - 20 + 10i + 25 \\ &= 0 \end{aligned}$$

so -2 + i is a root of the equation.

Since -2 + i is a root, -2 – i is also a root.
Therefore (z + 2 – i) and (z + 2 + i) are both factors.

Here the quartic equation has been factorised into the known quadratic factor and another quadratic factor. The PowerPoint presentation **Factorising quartics** demonstrates some different methods for doing this.

So a quadratic factor is $(z+2-\text{i})(z+2+\text{i}) = (z+2)^2 + 1$

$$= z^2 + 4z + 4 + 1$$

$$= z^2 + 4z + 5$$

$$z^4 + 2z^3 + 2z^2 + 10z + 25 = 0$$

$$(z^2 + 4z + 5)(z^2 - 2z + 5) = 0$$

The other roots are the roots of the quadratic equation $z^2 - 2z + 5 = 0$.
Using the quadratic formula:

$$z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4\text{i}}{2}$$

$$= 1 \pm 2\text{i}$$

The roots are -2 – i, -2 + i, 1 + 2i and 1 – 2i.